

REGULARITY OF POLYNOMIALS IN FREE VARIABLES

IAN CHARLESWORTH AND DIMITRI SHLYAKHTENKO

ABSTRACT. We show that the spectral measure of any non-commutative polynomial of a non-commutative n -tuple cannot have atoms if the free entropy dimension of that n -tuple is n (see also work of Mai, Speicher, and Weber). Under stronger assumptions on the n -tuple, we prove that the spectral measure is not singular, and measures of intervals surrounding any point may not decay slower than polynomially as a function of the interval's length.

1. INTRODUCTION.

It was shown in [SS13] that if y_1, \dots, y_n are free self-adjoint non-commutative random variables and P is any self-adjoint non-commutative polynomial in n indeterminates, then the spectral measure of $y = P(y_1, \dots, y_n)$ cannot have atoms, unless P is constant. This in particular implies that a polynomial of n semicircular elements cannot have atoms. When applied to random matrix theory, this shows that if Y_1, \dots, Y_n are independent Gaussian Random Matrices, then the eigenvalues of $P(Y_1, \dots, Y_n)$ exhibit a very weak form of repulsion: the expected proportion of the number of eigenvalues in any interval $[a, b]$ goes to zero with N as the size of the interval shrinks.

Statements of this kind can be considered to be part of the study of consequences of regularity assumptions on non-commutative transformations, which is of significant interest due to, for example, the results of [GS12]. Since [SS13], there have been important advances in this direction. Very recently, Figalli and Guionnet [GF14] have used transport maps to give a full picture of the behavior of eigenvalues of $P(Y_1, \dots, Y_n)$ under the assumption that P is close to the identity transformation. Finally, Mai, Speicher and Weber [MSW15] have been able to obtain a regularity result for $p(y_1, \dots, y_n)$ in absence of freeness assumption and beyond the perturbative regime. While initially requiring a stronger assumption, they were also able to give a proof of the following theorem using their methods after seeing an early version of our note; our proof appears in the next section.

Theorem. *Assume that Voiculescu's free entropy dimension [Voi94, Voi98] $\delta^*(y_1, \dots, y_n) = n$. Then for any self-adjoint non-constant non-commutative polynomial P , the spectral measure of $y = P(y_1, \dots, y_n)$ has no atoms. In particular, $\delta^*(y) = 1$.*

The assumption of the theorem is satisfied in each of the following cases:

- (1) If y_j are freely independent and each has non-atomic distribution (compare [SS13]). Indeed, in this case $\delta^*(y_j) = 1$ and so by free independence $\delta^*(y_1, \dots, y_n) = \sum \delta^*(y_j) = n$. In particular, this holds if y_1, \dots, y_n are a free semicircular family.
- (2) If $\Phi^*(y_1, \dots, y_n) < +\infty$.
- (3) If Voiculescu's microstates free entropy [Voi94] $\chi(y_1, \dots, y_n)$ or non-microstates free entropy [Voi98] $\chi^*(y_1, \dots, y_n)$ are finite, or if the microstates free entropy dimension $\delta(y_1, \dots, y_n) = n$. Indeed, in all of these cases $\delta^*(y_1, \dots, y_n) = n$ (making use of the inequality between microstates and non-microstates free entropy proved in [BCG03]).

It is not hard to see that the hypothesis of this theorem is optimal. Indeed, for a free n -tuple y_1, \dots, y_n in which the law of y_1 has a single atom of weight λ and the laws of y_2, \dots, y_n are non-atomic, one has $\delta^*(y_1, \dots, y_n) = n - \lambda^2$.

We also prove that if $\delta^*(y_1, \dots, y_n) = n$, then there can be no algebraic relations between convergent power series in y_1, \dots, y_n . This was proved under stronger assumptions by Dabrowski (see Lemma 37 in [Dab10]); a version for polynomials appears in [MSW15].

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Our proof uses ideas from L^2 -homology going back to the second author's joint work with Connes [CS05] to give an alternate proof of the key lemma in [MSW15]. The remainder of the proof is essentially the same as in [MSW15].

In Section 3, we demonstrate that if $y \in M$ has no atoms and is algebraic (i.e., has spectral measure with algebraic Cauchy transform), then $\chi(y) > -\infty$. The argument hinges on work of Anderson and Zeitouni in [AZ06], and in particular Theorem 2.9 of that paper, which controls the density of the spectral measure of such a variable y . This leads us to the following result:

Corollary. *Assume that y_1, \dots, y_n are free, algebraic, and $\chi(y_j) > -\infty$ for $1 \leq j \leq n$. Then if $y = P(y_1, \dots, y_n)$ with P a non-constant self-adjoint polynomial, $\chi(y) > -\infty$.*

In the final section of this paper, we show that under some stronger assumptions, such as the existence of a dual system, the spectral measure of $y = P(y_1, \dots, y_n)$ cannot be purely singular with respect to Lebesgue measure, and in certain cases such as when y is a monomial, must be absolutely continuous. Additionally, we demonstrate that under the same assumption of a dual system, the spectral measure of y must not decay at a rate slower than polynomially in the neighbourhood of any fixed point.

2. ABSENCE OF ATOMS AND ZERO DIVISORS.

2.1. Non-commutative polynomials, power series and difference quotients. If M is an algebra, $u, v \in M$ and T is an element of an M, M -bimodule, then we use the notation $(u \otimes v) \# T = uTv$. We view $M \otimes M$ as an M, M -bimodule as follows: for $a \otimes b \in M \otimes M$, $u, v \in M$, set

$$(u \otimes v) \# (a \otimes b) = ua \otimes bv.$$

We denote by $\mathbb{C}[X_1, \dots, X_n]$ the algebra of non-commuting polynomials in n indeterminates X_1, \dots, X_n . We denote by ∂_j Voiculescu's difference quotient derivations [Voi98] with values in $\mathbb{C}[X_1, \dots, X_n] \otimes \mathbb{C}[X_1, \dots, X_n]$, determined by $\partial_j X_i = \delta_{i=j} 1 \otimes 1$.

We denote by $\mathbb{C}\langle X_1, \dots, X_n; R \rangle$ the completion of $\mathbb{C}[X_1, \dots, X_n]$ in the norm

$$\left\| \sum_N \sum_{j_1, \dots, j_N=1}^n \alpha_{N;j_1, \dots, j_N} X_{j_1} \cdots X_{j_N} \right\|_R = \sum_N \sum_{j_1, \dots, j_N=1}^n |\alpha_{N;j_1, \dots, j_N}| R^N.$$

We will also write

$$\mathcal{A}[X_1, \dots, X_n; R] = \bigcup_{R' > R} \mathbb{C}\langle X_1, \dots, X_n; R' \rangle$$

for the algebra of power series with norm strictly bigger than R . Of course, if y_1, \dots, y_n are elements of any Banach algebra M and $\max_j \|y_j\| < R$, then there is a unique homomorphism from $\mathcal{A}[X_1, \dots, X_n; R]$ which sends X_j to y_j , $j = 1, \dots, n$. For $P \in \mathcal{A}[X_1, \dots, X_n; R]$, we write $P(y_1, \dots, y_n)$ for the image of P under this homomorphism.

It is not hard to see that ∂_j extends to an (unbounded) derivation

$$\partial_j : \mathcal{A}[X_1, \dots, X_n; R] \rightarrow \mathcal{A}[X_1, \dots, X_n; R] \hat{\otimes} \mathcal{A}[X_1, \dots, X_n; R],$$

where $\hat{\otimes}$ denotes the projective tensor product. Likewise, if B is any Banach bimodule over $\mathbb{C}[X_1, \dots, X_n]$ for which the right and left actions of X_j are bounded operators of norm at most R , then $\#$ extends to an action of $\mathcal{A}[X_1, \dots, X_n; R] \hat{\otimes} \mathcal{A}[X_1, \dots, X_n; R]$ on B . In particular, if Q is any element of the space $\mathcal{L}^1(L^2(M))$ of trace-class operators on M , then $P \mapsto \partial_j(P) \# Q$ extends to a map defined on $\mathcal{A}[X_1, \dots, X_n; R]$.

Let $\mathcal{N} : \mathcal{A}[X_1, \dots, X_n; R] \rightarrow \mathcal{A}[X_1, \dots, X_n; R]$ be given by $\mathcal{N}P = \sum_{j=1}^n \partial_j P \# X_j$. Thus \mathcal{N} is a kind of number operator, multiplying a monomial by its degree.

We note that the formula

$$(2.1) \quad \phi_t(P) = \sum_{k \geq 0} \mathcal{N}^k (2\pi i t)^k / k!$$

gives rise to an automorphism of $\mathcal{A}[X_1, \dots, X_n; R]$ which multiplies a monomial of degree d by $\exp(2\pi i t d)$.

2.2. Algebraic relations and Hochschild cycles. The following lemma shows that algebraic relations in the algebra generated by y_1, \dots, y_n produce Hochschild cycles:

Lemma 1. *Suppose that $P \in \mathcal{A}[X_1, \dots, X_n; R]$ and suppose that for some self-adjoint elements $y_1, \dots, y_n \in (M, \tau)$ with $\max \|y_j\| < R$ and $u, v \in M$, $(u \otimes v) \# (P(y_1, \dots, y_n) \otimes 1 - 1 \otimes P(y_1, \dots, y_n)) = 0$. Let $T_i = (\partial_i P)(y_1, \dots, y_n) \in M \otimes M$. Then*

$$\sum (u \otimes v) \# T_i \# (y_i \otimes 1 - 1 \otimes y_i) = 0.$$

Thus if we put for $y, y' \in M$, $Jy^*J(a \otimes b)J(y')^*J = ay \otimes y'b$, the identity

$$\sum [uT_iv, Jy_i^*J] = 0$$

holds.

Proof. Note that

$$(u \otimes v) \# \sum T_i (y_i \otimes 1 - 1 \otimes y_i) = (u \otimes v) \# (P(y_1, \dots, y_n) \otimes 1 - 1 \otimes P(y_1, \dots, y_n)) = 0,$$

which gives the first equation. The second equation follows from the definition of the action of Jy_i^*J and commutation of the variables u, v with the variables Jy_j^*J , $j = 1, \dots, n$. \square

Let $\delta^*(y_1, \dots, y_n)$ be Voiculescu's non-microstates free entropy dimension. Denote by $FR = FR(L^2(M))$ the space of finite-rank operators on $L^2(M)$. Let $\Delta(y_1, \dots, y_n)$ be the quantity introduced in [CS05]:

$$\Delta(y_1, \dots, y_n) = n - \dim_{M \bar{\otimes} M} \overline{\{(T_1, \dots, T_n) \in FR(L^2(M)) : \sum [T_j, Jy_j^*J] = 0\}},$$

where the closure is taken in the Hilbert-Schmidt norm.

Let us denote by $\mathcal{L}^1 = \mathcal{L}^1(L^2(M))$ the space of trace-class operators on $L^2(M)$. Let

$$\Delta_1(y_1, \dots, y_n) = n - \dim_{M \bar{\otimes} M} \overline{\{(T_1, \dots, T_n) \in \mathcal{L}^1(L^2(M)) : \sum [T_j, Jy_j^*J] = 0\}},$$

where once again the closure is taken in the Hilbert-Schmidt norm. Clearly, $\Delta_1 \leq \Delta$. By [CS05, Lemma 4.1 and Theorem 4.4] we have inequalities

$$\delta^*(y_1, \dots, y_n) \leq \Delta(y_1, \dots, y_n) \leq n.$$

The same theorem (with exactly the same proof) shows that also $\delta^*(y_1, \dots, y_n) \leq \Delta_1(y_1, \dots, y_n)$ (indeed, the only change is in the Lemma 4.2, which obviously works if we replace FR by \mathcal{L}^1). Thus

$$\delta^*(y_1, \dots, y_n) \leq \Delta_1(y_1, \dots, y_n) \leq \Delta(y_1, \dots, y_n) \leq n.$$

In particular, if $\delta^*(y_1, \dots, y_n) = n$, then $\Delta(y_1, \dots, y_n) = n$, which implies that

$$\overline{\{(T_1, \dots, T_n) \in \mathcal{L}^1(L^2(M)) : \sum [T_j, Jy_j^*J] = 0\}} = \{0\}.$$

We record this as:

Corollary 2. [CS05] *Let $M = W^*(y_1, \dots, y_n)$. Assume that $\delta^*(y_1, \dots, y_n) = n$. Then the only trace-class operators $R_j \in \mathcal{L}^1(L^2(M, \tau))$, $j = 1, \dots, n$, that satisfy*

$$\sum [R_i, Jy_i^*J] = 0$$

are $R_1 = R_2 = \dots = R_n = 0$.

2.3. Absence of zero divisors and algebraic relations.

Theorem 3. *Let $y_1, \dots, y_n \in M$ be self-adjoint elements for which $\delta^*(y_1, \dots, y_n) = n$. Assume that for some $P \in \mathbb{C}[X_1, \dots, X_n]$ and some projection $0 \neq p \in W^*(y_1, \dots, y_n)$, $P(y_1, \dots, y_n) \cdot p = 0$. Then $P = 0$.*

Proof. Assume for contradiction that $P \neq 0$; obviously, P is not constant. We may further assume that P is the smallest degree non-constant polynomial with the property that $P(y_1, \dots, y_n) \cdot p = 0$.

Because M is a tracial von Neumann algebra, the left and right support projections of $P(y_1, \dots, y_n)$ are equal; thus for some projection q_1 with the same trace as p , $q_1 P(y_1, \dots, y_n) = 0$.

We apply Lemma 1 with $u = q_1$, $v = p$ to conclude that if we set $R_j = (u \otimes 1 + 1 \otimes v) \# (\partial_j P)(y_1, \dots, y_n)$, then $\sum [q_1 R_j P, J y_j^* J] = 0$. Then Corollary 2 implies that $R_j = 0$ for all j ; in other words,

$$(2.2) \quad (q_1 \otimes 1 + 1 \otimes p) \partial_j(P)(y_1, \dots, y_n) = 0.$$

Let

$$\Delta_{j, q_1}(P) = (\tau \otimes 1) ((q_1 \otimes 1) \# \partial_j(P)) \in \mathbb{C}[X_1, \dots, X_n].$$

Then $\Delta_{j, q_1}(P)(y_1, \dots, y_n)p = 0$, but the degree of $\Delta_{j, q_1}(P)$ is smaller than that of P . It follows that $\Delta_{j, q_1}P = 0$ or that P is linear. In the former case, let $X_{i_1} \dots X_{i_s}$ be a highest-degree term in P with a nonzero coefficient, α . Then applying our proof recursively, we obtain projections q_1, q_2, \dots, q_s of trace equal to p , so that $(\Delta_{i_s, q_s} \circ \dots \circ \Delta_{i_1, q_1})(P) = \alpha \prod \tau(q_j) = \alpha \tau(p)^s$, contradicting $\Delta_{i_1, q_1}(P) = 0$. If P is linear, then $\Delta_{j, q_1}(P)$ does not depend on X_1, \dots, X_n and so $\Delta_{j, q_1}(P)(y_1, \dots, y_n) = 0$ implies that P is constant. \square

Theorem 4. *Let $y_1, \dots, y_n \in M$ be self-adjoint elements satisfying $\max_j \|y_j\| < R$ and $\delta^*(y_1, \dots, y_n) = n$. Assume that for some $P \in \mathcal{A}[X_1, \dots, X_n; R]$, $P(y_1, \dots, y_n) = 0$. Then $P = 0$.*

Proof. Assume for contradiction that $P \neq 0$. Obviously, P is not constant.

We apply Lemma 1 with $u = v = 1$: if $R_j = (\partial_j P)(y_1, \dots, y_n)$, then $\sum [R_j, J y_j^* J] = 0$. Thus Corollary 2 implies that $R_j = 0$ for all j ; in other words,

$$\partial_j(P)(y_1, \dots, y_n) = 0$$

It follows that also $\mathcal{N}P = \sum_j \partial_j P \# X_j$ satisfies

$$\mathcal{N}P(y_1, \dots, y_n) = 0.$$

Hence if ϕ_t is given by (2.1), then

$$\phi_t(P)(y_1, \dots, y_n) = 0.$$

Thus

$$\int_0^1 \exp(-2\pi i m t) \phi_t(P)(y_1, \dots, y_n) dt = 0, \quad \forall m > 0.$$

Thus if we write $P^{(m)}$ for the sum of degree m terms of P , then $P^{(m)}(y_1, \dots, y_n) = 0$, for each $m > 0$. But then we may use Theorem 3 with $p = 1$ to conclude that $P^{(m)} = 0$ for all $m > 0$ and that $P = 0$. \square

2.4. Further questions. Assume now that $z_1, \dots, z_m \in M$ are self-adjoint, but make no assumptions on whether they generate M or not. Put

$$\Delta_M(z_1, \dots, z_m) = m - \dim_{M \otimes M^o} \overline{\{(T_1, \dots, T_m) \in FR(L^2(M))^m : \sum [T_j, J z_j^* J] = 0\}}^{HS}.$$

Conjecture 5. *Let $y_1, \dots, y_n \in M$ be self-adjoint elements so that $M = W^*(y_1, \dots, y_n)$.*

(a) *Let $P_1, \dots, P_m \in \mathbb{C}[X_1, \dots, X_n]$ be polynomials. Let $z_j = P_j(y_1, \dots, y_n)$, $j = 1, \dots, m$. Assume finally that $M = W^*(y_1, \dots, y_n)$. Assume that $\Delta_M(y_1, \dots, y_n) = n$. Then exactly one of the following statements holds:*

- (i) $\Delta_M(z_1, \dots, z_m) \leq m - 1$; or
- (ii) $\Delta_M(z_1, \dots, z_m) = m$.

Moreover, if (b) holds and $\chi^*(y_1, \dots, y_n) > -\infty$ then also $\chi^*(z_1, \dots, z_m) > -\infty$.

(b) *Let $N \geq 1$ be fixed, and let $P_{ij} \in \mathbb{C}[X_1, \dots, X_n]$, $1 \leq i, j \leq N$. Let $y = (P_{ij}(y_1, \dots, y_n))_{ij} \in M_{N \times N}(M)$, and assume that $y = y^*$. Then any atom of the spectral measure of y must have normalized trace in the set $\{0, 1/N, \dots, N/N\}$.*

Part (b) above is known to hold in the special case that y_1, \dots, y_n are a free semi-circular family; see [SS13].

3. ALGEBRAICITY AND FREE ENTROPY.

Lemma 6. *Suppose that $y = y^* \in (M, \tau)$ is an element of a von Neumann probability space. Further assume that the spectral distribution of y is Lebesgue absolutely continuous, with density f . Lastly, suppose that $f \in L^p(\mathbb{R})$ for some $p > 1$. Then $\chi(y) > -\infty$.*

Proof. Using interpolation and the fact that $\|f\|_1 = 1$, we may assume that $p < 2$. For a single variable, we know by [Voi93] that the free entropy satisfies

$$\chi(x) = \iint \log |s - t| d\mu(s) d\mu(t) + C,$$

for C constant, so it suffices to bound the integral. Let $L(t) := \chi_{(-4M, 4M)} \log |t|$, where M is large enough that the support of μ is contained in $(-M, M)$. Then we have for $s \in \mathbb{R}$,

$$f(s) \int f(t) \log |s - t| dt = f(s) \int f(t) L(s - t) dt = f(s)(f \star L)(s).$$

We compute:

$$\begin{aligned} \left| \iint \log |t - s| d\mu(t) d\mu(s) \right| &= \left| \int f(s) \int f(t) \log |s - t| dt ds \right| \\ &= \left| \int f(s)(f \star L)(s) ds \right| \\ &\leq \|f \cdot (f \star L)\|_1 \\ &\leq \|f\|_p \|f \star L\|_q, \end{aligned}$$

where $1 = \frac{1}{p} + \frac{1}{q}$, by Hölder's inequality; note that $q > 2$ since $p < 2$. It suffices, now, to show that $f \star L \in L^q(\mathbb{R})$; for this, we appeal to Young's inequality. Indeed, if $s = \frac{q}{2} > 1$, then $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{s}$ and so we have

$$\|f \star L\|_q \leq \|f\|_p \|L\|_s.$$

Yet $\|L\|_s < \infty$ for any $1 \leq s < \infty$, so we conclude $|\chi(x)| < \infty$. \square

Suppose that $y = y^* \in M$ is algebraic, in the sense that its spectral measure μ has algebraic Cauchy transform. Assume further that μ has no atoms. It was shown in [AZ06, Theorem 2.9] that such a measure μ has density f which fails to exist at at most finitely many points, and for some $d > 0$ and every $a \in \mathbb{R}$ satisfies

$$\lim_{x \rightarrow a} (x - a)^{1-d} f(x) < \infty.$$

In particular, if $1 < p < (1 - d)^{-1}$, we have $f \in L^p(\mathbb{R})$ and so Lemma 6 leads us to conclude $\chi(y) > -\infty$.

Corollary 7. *Assume that y_1, \dots, y_n are free, algebraic, and $\chi(y_j) > -\infty$ for $1 \leq j \leq n$. Then if $y = P(y_1, \dots, y_n)$ with P a non-constant polynomial, $\chi(y) > -\infty$.*

Proof. It was shown by Anderson in [A14] that the freeness and algebraicity of y_1, \dots, y_n is enough to guarantee that y is algebraic. Freeness and the fact that $\chi(y_j) > -\infty$ together with Theorem 3 allow us to conclude that the spectral measure of y has no atoms. Following the above discussion, we conclude $\chi(y) > -\infty$. \square

It is tempting to conjecture that if y_1, \dots, y_n satisfy $\chi(y_1, \dots, y_n) > -\infty$ (where we use Voiculescu's microstates free entropy, [Voi94]) and $y = p(y_1, \dots, y_n)$ is a non-constant polynomial, then also $\chi(y) > -\infty$. We point out that this is true assuming that p is sufficiently close to a polynomial of degree 1. For a non-commutative polynomial p let us write a_{i_1, \dots, i_n} for the coefficient of the monomial $y_{i_1} \cdots y_{i_n}$ in p :

Proposition 8. Assume that $\chi(y_1, \dots, y_n) > -\infty$. Fix d and n . Then there exists an $\epsilon > 0$ so that if $p(y_1, \dots, y_n)$ is a self-adjoint polynomial of degree d for which

$$\sum_{k \geq 2} \sum_{i_1, \dots, i_k=1}^n |a_{i_1, \dots, i_k}|^2 < \epsilon \sum_j |a_j|^2$$

and $y = p(y_1, \dots, y_n)$, then

$$\chi(y) > -\infty.$$

Proof. We note first that we can assume that p has no constant term, since this has no effect on the free entropy of y . Let $\delta = \sum_j |a_j|^2$. We note that by Voiculescu's change of variable formula, $\chi(y_1, \dots, y_n)$ is unchanged if we replace y_j by $\sum_i T_{ij} y_i$, where (T_{ij}) is any orthogonal matrix. We may thus assume that $a_j = 0$ unless $j = 1$ and that $|a_1|^2 = \delta$. By multiplying p by δ^{-1} (which changes the free entropy of y by an finite additive constant) we may assume that $a_1 = 1$ and $\sum_{i_1, \dots, i_k=1}^n |a_{i_1, \dots, i_k}|^2 < \epsilon$.

Next, we set $q_1(y_1, \dots, y_n) = p$ and let $q_j(y_1, \dots, y_n) = y_j$ for $j = 2, \dots, n$. Then by the implicit function theorem for non-commutative power series (see e.g. [GS12]) we know that as long as ϵ is sufficiently small, there exist non-commutative power series f_1, \dots, f_n so that

$$\begin{aligned} t_j &= q_j(f_1(t_1, \dots, t_n), \dots, f_n(t_1, \dots, t_n)) \\ s_j &= f_j(q_1(s_1, \dots, s_n), \dots, q_n(s_1, \dots, s_n)) \end{aligned}$$

for all j and any operators t_1, \dots, t_n and s_1, \dots, s_n of norm at most $1.1 \max_j \|y_j\|$. Voiculescu's change of variables formula for free entropy [Voi94] implies that

$$\chi(q_1(y_1, \dots, y_n), \dots, q_n(y_1, \dots, y_n)) > -\infty.$$

But then

$$-\infty < \chi(y, y_2, \dots, y_n) \leq \chi(y) + \chi(y_2) + \dots + \chi(y_n)$$

which implies that $\chi(y) > -\infty$. \square

4. PROPERTIES OF SPECTRAL MEASURES.

4.1. Non-singularity. For A an algebra, we denote the flip operation on A as follows: given $a, b \in A$, set $(a \otimes b)^\sigma = b \otimes a$, and extend this linearly to $A \otimes A$. We also maintain the notation $(a \otimes b) \# z = azb$ for $a, b, z \in A$.

Lemma 9. Let $x \in B(H)$ be a self-adjoint operator, and assume that the spectral measure of x is not Lebesgue absolutely continuous. Then there exists a sequence T_n of finite-rank operators satisfying the following properties: (i) $0 \leq T_n \leq 1$; (ii) for a nonzero spectral projection p of x , $T_n \rightarrow p$ weakly; (iii) $\|[T_n, x]\|_1 \rightarrow 0$. Moreover, if the absolutely continuous component of the spectral measure of x is zero, then p can be chosen to be 1.

Proof. Let p be a nonzero spectral projection of x corresponding to a subset of \mathbb{R} having Lebesgue measure zero. By replacing x with $p x p$, we may assume that x has singular spectral measure. In this case, by a result of Voiculescu (see Theorem 4.5 in [Voi79]; note that as remarked at the bottom of page 5 of that article, $\mathcal{C}_1^- = \mathcal{C}_1$ and so $k_1^- = k_1$; in this particular case the result essentially goes back to Kato [Kat66]), there exist $0 \leq T_n \leq 1_{pH}$ with the property that $T_n \rightarrow 1_{pH}$ weakly in $B(pH)$ and $\|[x, T_n]\|_1 \rightarrow 0$. \square

Lemma 10. Let $\text{Alg}(y_1, \dots, y_n) = A$ be a $*$ -algebra with $y_i = y_i^*$, $y \in A$ a polynomial, and τ a faithful trace on A . Suppose that y_1, \dots, y_n are algebraically free, and for each $1 \leq i \leq n$ let $\partial_i : A \rightarrow A \otimes A$ be the derivation given by $\partial_i(y_j) = \delta_{j=i} 1 \otimes 1$. Then $y \in \text{Alg}(y_1, \dots, \hat{y}_i, \dots, y_n)$ if and only if

$$(\partial_i y)^\sigma \# y^* = 0.$$

Moreover, if $(\partial_i y)^\sigma \# y^* = 0$ for every $1 \leq i \leq n$, then $y \in \mathbb{C}$.

Proof. One direction is immediate as $\text{Alg}(y_1, \dots, \hat{y}_i, \dots, y_n) \subseteq \ker \partial_i$.

Let $\mathcal{N}_i : A \rightarrow A$ be the number operator associated to y_i , the linear map which multiplies each monomial by its y_i -degree. Observe that

$$(\partial_i y) \# y_i = \mathcal{N}_i(y),$$

as each monomial m in y contributes $\sum_{m=ay_ib}(a \otimes b) \# y_i = \mathcal{N}_i(m)$.

Suppose, then, that $(\partial_i y)^\sigma \# y^* = 0$, so $y_i(\partial_i y)^\sigma \# y^* = 0$ as well. Let $\varphi_\lambda : A \rightarrow A$ be the algebra homomorphism given by $\varphi_\lambda(y_i) = \lambda y_i$, $\varphi_\lambda(y_j) = y_j$ for $j \neq i$, which exists as y_1, \dots, y_n are algebraically free. We compute the following:

$$0 = \tau \circ \varphi_\lambda(0) = \tau \circ \varphi_\lambda(y_i(\partial_i y)^\sigma \# y^*) = \tau \circ \varphi_\lambda(y^*(\partial_i y) \# y_i) = \tau \circ \varphi_\lambda(y^* \mathcal{N}_i(y)).$$

Now, suppose $\deg_{y_i}(y) = d$, and take $x, z \in A$ so that x is y_i -homogeneous of degree d , $\deg_{y_i}(z) < d$, and $y = x + z$. Then:

$$0 = \tau \circ \varphi_\lambda(y^* \mathcal{N}_i(y)) = \tau \circ \varphi_\lambda(dx^*x) + \tau \circ \varphi_\lambda(z^* \mathcal{N}_i(y) + x^* \mathcal{N}_i(z)) = d\lambda^{2d}\tau(x^*x) + O(\lambda^{2d-1}).$$

Thus $d\lambda^{2d}\tau(x^*x) = 0$, and as τ is faithful, either $x = 0$ (in which case $y = 0$) or $d = 0$ (in which case $\deg_{y_i}(y) = 0$). Either way, $y \in \text{Alg}(y_1, \dots, \hat{y}_i, \dots, y_n)$.

Repeated application of the above yields the final claim. \square

Theorem 11. *Let $y = y^* \in \text{Alg}(y_1, \dots, y_n) \subset W^*(y_1, \dots, y_m) = M$ be a non-constant polynomial in a W^* -probability space and assume that there exists a dual system to y_1, \dots, y_m , i.e., operators $R_j \in B(L^2(M))$, $1 \leq j \leq m$ so that $[R_j, y_k] = \delta_{j=k}P_1$, where P_1 is the orthogonal projection onto $1 \in L^2(M)$. Then the spectral measure of y is not singular with respect to Lebesgue measure.*

Proof. Assume to the contrary that the spectral measure of y is singular with respect to Lebesgue measure. By Lemma 9, we may choose $0 \leq T_n \leq 1$ finite rank operators with $T_n \rightarrow 1$ weakly and $\|[T_n, y]\|_1 \rightarrow 0$.

Fix $x \in M$. We compute as follows:

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \|Jx^*JR_iy\|_\infty \|[T_n, y]\|_1 \\ &\geq \lim_{n \rightarrow \infty} \left| \text{Tr}(Jx^*JR_iy[T_n, y]) \right| \\ &= \lim_{n \rightarrow \infty} \left| \text{Tr}(Jx^*J[y, R_i]yT_n) \right| \\ &= \lim_{n \rightarrow \infty} \left| \text{Tr}(Jx^*J((\partial_i y) \# P_1)yT_n) \right| \\ &= \left| \text{Tr}(Jx^*JP_1((\partial_i y)^\sigma \# y)) \right| \\ &= \left| \tau(((\partial_i y)^\sigma \# y)x) \right| \end{aligned}$$

Here we are able to pass to the limit of T_n as P_1 is finite rank.

We conclude that $(\partial_i y)^\sigma \# y = 0$ as τ is faithful and as x was arbitrary. From Lemma 10, we see that y is constant. \square

For certain polynomials, we can make a stronger statement: that the spectral measure is in fact Lebesgue absolutely continuous. To do so, we will need the following modification of Lemma 10.

Lemma 12. *Let $y = y^* \in \text{Alg}(y_1, \dots, y_n) \subset W^*(y_1, \dots, y_m) = M$ be a polynomial in a tracial von Neumann probability space, suppose y_1, \dots, y_m are algebraically free and have full free entropy dimension, and take $\partial_i : A \rightarrow A \otimes A$ as before. Fix $1 \leq i \leq n$, and assume that for some $p \in \mathcal{P}(W^*(y))$ a non-zero spectral projection of y , we have*

$$(\partial_i y)^\sigma \# (py) = 0.$$

Finally, suppose that y is an eigenvector of the number operator \mathcal{N}_i , i.e., $\mathcal{N}_i(y) = \partial_i y \# y_i = \lambda y$. Then $\lambda = 0$ and $y \in \text{Alg}(y_1, \dots, \hat{y}_i, \dots, y_n)$.

Moreover, if $(\partial_i y)^\sigma \# (py) = 0$ for each $1 \leq i \leq n$ and y is an eigenvector of each $\mathcal{N}_1, \dots, \mathcal{N}_n$, then $y \in \mathbb{C}$.

Proof. As in the proof of Lemma 10, we compute:

$$0 = \tau(y_i(\partial_i y)^\sigma \# (py)) = \tau(py(\partial_i y) \# y_i) = \tau(py\mathcal{N}_i(y)) = \lambda\tau(pyyp).$$

If $\lambda \neq 0$, it follows that $pyyp = 0$, hence $py = 0$. But then by Theorem 3 we have that y is constant. Thus in either case, $y \in \text{Alg}(y_1, \dots, \hat{y}_i, \dots, y_n)$. \square

Theorem 13. *Let $y = y^* \in \text{Alg}(y_1, \dots, y_m) \subset W^*(y_1, \dots, y_m) = M$ be a non-constant polynomial in a W^* -probability space and assume that there exists a dual system to y_1, \dots, y_m , i.e., operators $R_j \in B(L^2(M))$, $1 \leq j \leq m$ so that $[R_j, y_k] = \delta_{j=k} P_1$, where P_1 is the orthogonal projection onto $1 \in L^2(M)$. Suppose further that for each j , with \mathcal{N}_j the number operator corresponding to y_j , we have $\mathcal{N}_j(y) = \lambda_j y$. Then the spectral measure of y is Lebesgue absolutely continuous.*

Note that the hypothesis that y is an eigenvector of each \mathcal{N}_j is satisfied when y is homogeneous in each y_j , and in particular, whenever y is a monomial.

Proof. We proceed much as in the proof of Proposition 11. Assume to the contrary that the spectral measure of y is not Lebesgue absolutely continuous. As before, we apply Lemma 9 to find $0 \leq T_n \leq 1$ finite rank operators with $T_n \rightarrow p$ weakly for p a spectral projection of y , and $\|[T_n, y]\|_1 \rightarrow 0$.

Fix $x \in M$. We compute as follows:

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \|Jx^* J R_i y\|_\infty \|[T_n, y]\|_1 \\ &\geq \lim_{n \rightarrow \infty} \left| \text{Tr}(Jx^* J R_i y [T_n, y]) \right| \\ &= \lim_{n \rightarrow \infty} \left| \text{Tr}(Jx^* J [y, R_i] y T_n) \right| \\ &= \lim_{n \rightarrow \infty} \left| \text{Tr}(Jx^* J ((\partial_i y) \# P_1) y T_n) \right| \\ &= \left| \text{Tr}(Jx^* J P_1 ((\partial_i y)^\sigma \# (yp))) \right| \\ &= \left| \tau(((\partial_i y)^\sigma \# (yp))x) \right| \end{aligned}$$

Here we are able to pass to the limit of T_n as P_1 is finite rank.

We conclude that $(\partial_i y)^\sigma \# (yp) = 0$ as τ is faithful and x was arbitrary. Our stronger assumptions on y now allow us to apply Lemma 12, and we see y is constant. \square

A related statement is the following. Denote by \vec{T} an m -tuple of operators $\vec{T} = (T_1, \dots, T_m)$. Write $\vec{T} \geq \vec{T}'$ if $T_j \geq T'_j$ for all $j = 1, \dots, m$. Write \mathcal{R}_+^m for the set of all m -tuples \vec{T} satisfying $0 \leq T_j \leq 1$.

Proposition 14. *Assume that $y_1, \dots, y_m \in M$ admits a dual system R_1, \dots, R_m . Then*

$$\liminf_{\vec{T} \in \mathcal{R}_+^m} \left\| \sum [T_j, y_j] \right\|_1 > 0.$$

Proof. Assume for contradiction that we can find $\vec{T}^n \in \mathcal{R}_+^m$ satisfying $T_j \rightarrow 1$ and $\|\sum [T_j, y_j]\|_1 \rightarrow 0$. Then

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \text{Tr} \left(\sum_j R_i [T_j^n, y_j] \right) \\ &= \lim_{n \rightarrow \infty} \text{Tr} ([y_j, R_i] T_j^n) \\ &= \lim_{n \rightarrow \infty} \text{Tr} (P_1 T_i^n) \\ &= \text{Tr} (P_1) = 1 \end{aligned}$$

which is a contradiction. \square

4.2. Decay of measure.

Lemma 15. *Suppose that $y_1, \dots, y_m \in W^*(y_1, \dots, y_m) = M$ have a dual system of operators $R_1, \dots, R_m \in B(L^2(M))$. Assume that for some $P_n \in \text{Alg}(y_1, \dots, y_m)$, there exists a sequence of projections $f_n \in M$ so that $\|f_n\|_\infty = 1$, $\|f_n\|_2 \rightarrow 0$, $\|P_n f_n\|_2 / \|f_n\|_2^k \rightarrow 0$ for all $k \geq 0$; moreover, assume that $p = \sup_n \deg P_n < +\infty$, and for each monomial $y_{i_1} \cdots y_{i_p}$, its coefficient $a_{i_1 \dots i_p}^{(n)}$ in P_n is uniformly bounded in n . Then all coefficients of monomials of degree p in P_n must converge to zero.*

Proof. Let $P_n = u_n b_n$ be the polar decomposition of P_n , with u_n a unitary and $b_n = (P_n^* P_n)^{1/2}$. Then, letting $g_n = u_n f_n u_n^*$, we obtain another projection g_n with $\|g_n\|_\infty = 1$, $\|g_n\|_2 = \|f_n\|_2$ and

$$\|P_n f_n\|_2 = \|u_n b_n f_n\|_2 = \|f_n b_n u_n^*\|_2 = \|f_n b_n\|_2 = \|f_n u_n^* u_n b_n\|_2 = \|u_n f_n u_n^* u_n b_n\|_2 = \|g_n P_n\|_2.$$

Thus, in particular, $\|g_n P_n\|_2 / \|g_n\|_2^k \rightarrow 0$, for any $k \geq 0$.

Then we have:

$$(g_n \otimes f_n) \# (P_n \otimes 1 - 1 \otimes P_n) = (g_n \otimes f_n) \# \left(\sum_i \partial_i P_n \right) \# (y_i \otimes 1 - 1 \otimes y_i),$$

so that if we set $T_i^n = (g_n \otimes f_n) \# \partial_i P_n$, then

$$\left\| \sum [T_i^n, y_i] \right\|_1 = \|g_n P_n \otimes f_n - g_n \otimes P_n f_n\|_1 \leq \|P_n f_n\|_2 \|g_n\|_2 + \|g_n P_n\|_2 \|f_n\|_2 = 2 \|P_n f_n\|_2 \|f_n\|_2.$$

Thus for any $D \in B(H)$,

$$\left| \sum \text{Tr}(T_i^n [y_i, D]) \right| = \left| \text{Tr} \left(\sum [T_i^n, y_i] D \right) \right| \leq \left\| \sum [T_i^n, y_i] \right\|_1 \|D\|_\infty \leq 2 \|P_n f_n\|_2 \|f_n\|_2 \|D\|_\infty.$$

Note that

$$\|T_i^{n\dagger}\|_\pi \leq \|f_n \otimes g_n\|_\pi \|(\partial_i P_n)^\dagger\|_\pi \leq C$$

where $C = \sup_n \max_i \|(\partial_i P_n)^\dagger\|_\pi < \infty$ by our assumptions of the uniform boundedness of both the degree of P_n and the coefficients of all monomials in P_n . Let $D^{(n)} = T_j^{n\dagger\sigma} \# R_j$. Then $\|D^{(n)}\| \leq \|T_j^{n\dagger}\|_\pi \|R_j\|_\infty = C \|R_j\|_\infty$. Moreover, $[y_i, D^{(n)}] = \delta_{i=j} T_i^{n\dagger}$. From this one gets that

$$\|T_j^n\|_2^2 = |\langle T_j^n, T_j^n \rangle| = \left| \sum_i \text{Tr}(T_i^n [y_i, D]) \right| \leq 2C \|R_j\|_\infty \|P_n f_n\|_2 \|f_n\|_2.$$

Thus

$$\|T_j^n\|_2 \leq \sqrt{2 \|R_j\|_\infty C} \|P_n f_n\|_2^{1/2} \|f_n\|_2^{1/2}.$$

Let now

$$\Delta_j P_n = \tau(g_n)^{-1} (\tau \otimes 1) ((g_n \otimes 1) \# \partial_j P_n).$$

Then

$$\deg \Delta_j P_n < \deg P_n$$

and coefficients of all monomials in $\Delta_j P_n$ are uniformly bounded. Moreover,

$$\begin{aligned} \|\Delta_j P_n f\|_2 &= \tau(g_n)^{-1} \|\tau \otimes 1 ((g_n \otimes f_n) \# \partial_j P_n)\|_2 \\ &\leq \tau(g_n)^{-1} \|(g_n \otimes f_n) \# \partial_j P_n\|_2 \\ &\leq \tau(g_n)^{-1} \|T_j^n\|_2 \\ &\leq \sqrt{2 \|R_j\|_\infty C} \|P_n f_n\|_2^{1/2} \|f_n\|_2^{1/2}. \end{aligned}$$

It follows that

$$\|\Delta_j P_n f_n\|_2 / \|f_n\|_2^k \leq \sqrt{2 \|R_j\|_\infty C} \cdot \|f_n\|_2^{1/2} \cdot \sqrt{\|P_n f_n\|_2 / \|f_n\|_2^{2k}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus $\Delta_i P_n$ once again satisfies the conditions of the Lemma. Iterating, so does $\Delta_{i_1} \Delta_{i_2} \cdots \Delta_{i_p} P_n$, for any sequence i_1, \dots, i_p .

Let $p = \max_n \deg P_n$, and let $a_{i_1 \dots i_p}^{(n)}$ be the coefficient of the monomial $y_{i_1} \cdots y_{i_p}$ in P_n . Then

$$a_{i_1 \dots i_p}^{(n)} = \Delta_{i_1} \cdots \Delta_{i_p} P_n,$$

which means that

$$|a_{i_1 \dots i_p}^{(n)}| = \|a_{i_1 \dots i_p}^{(n)} f_n\|_2 / \|f_n\|_2 \rightarrow 0,$$

as claimed. \square

Lemma 16. Let $P \in \text{Alg}(y_1, \dots, y_m)$ where again y_1, \dots, y_m admit a dual system. Suppose that there exists a sequence of projections $f_n \in M$ and $t \in \mathbb{C}$ so that $\|f_n\|_\infty = 1$, $\|f_n\|_2 \rightarrow 0$, $\|P f_n - t f_n\|_2 / \|f_n\|_2^k \rightarrow 0$ for all $k \geq 0$. Then P is constant.

Proof. By replacing P with $P - t1$ we may assume that $t = 0$. Let $P_n = P$ be the constant sequence. Then the coefficient of any monomial of maximal degree must be zero by Lemma 15. This means that $P = 0$. \square

Theorem 17. *Let $y = y^* \in \text{Alg}(y_1, \dots, y_n)$ be non-constant, where y_1, \dots, y_n admit a dual system, and let μ be the spectral measure of y . Then for every $t \in \mathbb{R}$ there is some $\alpha > 0$ so that $\mu([t, t + \epsilon]) \leq \epsilon^\alpha$ for all $\epsilon > 0$ small enough.*

Proof. We will say that μ satisfies condition $P(t, \alpha)$, $\alpha > 0$ if the inequality

$$\mu[t, t + \epsilon] \leq \epsilon^\alpha$$

holds for sufficiently small ϵ . Equivalently, we require that

$$\log \mu[t, t + \epsilon] \leq \alpha \log \epsilon$$

for sufficiently small ϵ .

Fix $t \in \mathbb{R}$, and assume to the contrary $P(t, \alpha)$ does not hold for any $\alpha > 0$. Then there exists a sequence $\epsilon_n \rightarrow 0$ with the property that

$$\frac{\log \mu[t, t + \epsilon_n]}{\log \epsilon_n} \rightarrow 0.$$

Put $\lambda_n = \mu[t, t + \epsilon_n]^{1/2}$. Then $\frac{\log \lambda_n}{\log \epsilon_n} \rightarrow 0$. Thus

$$\frac{(1 - k) \log \lambda_n + \log \epsilon_n}{\log \epsilon_n} \rightarrow 1.$$

Thus

$$(1 - k) \log \lambda_n + \log \epsilon_n \rightarrow -\infty.$$

Exponentiating, we get

$$\epsilon_n \lambda_n^{1-k} \rightarrow 0$$

Let $f_n = \chi_{[t, t + \epsilon]}(y)$. Then $\|f_n\|_\infty = 1$ and $\|f_n\|_2 = \lambda_n$. It follows that

$$\|yf_n - tf_n\|_2 \lesssim \epsilon_n \lambda_n$$

and so

$$\|yf_n - tf_n\|_2 / \|f_n\|_2^k \lesssim \epsilon_n \lambda_n^{1-k} \rightarrow 0.$$

Lemma 16 now implies that y is constant, a contradiction. \square

REFERENCES

- [A14] G. W. Anderson, *Preservation of algebraicity in free probability*, Preprint, 2014, arXiv:1406.6664.
- [AZ06] G. W. Anderson and O. Zeitouni, *A law of large numbers for finite-range dependent random matrices*, Comm. Pure App. Math, Vol. **61** (2008), 1118D1154.
- [BCG03] P. Biane, M. Capitaine and A. Guionnet, *Large deviation bounds for matrix Brownian motion*, Invent. Math., **152** (2003), 433–459.
- [CS05] A. Connes and D. Shlyakhtenko, *L^2 -homology for von Neumann algebras*, J. Reine Angew. Math., **586** (2005) 125–168.
- [Dab10] Y. Dabrowski, *A free stochastic partial differential equation*, Annal. IHP **50**, 2014, 1404–1455.
- [GF14] A. Figalli and A. Guionnet, *Universality in several-matrix models via approximate transport maps*, Preprint, 2014, arXiv:1407.2759
- [GS12] A. Guionnet and D. Shlyakhtenko, *Free monotone transport*, Invent. Math. **197** (2014), 613–661.
- [Kat66] T. Kato, *Perturbation theory of linear operators*, Springer, 1966.
- [MSW15] , T. Mai, R. Speicher and M. Weber, *Absence of algebraic relations and of zero divisors under the assumption of full non-microstates free entropy dimension*, Preprint, 2015.
- [SS13] D. Shlyakhtenko and P. Skoufranis, *Freely independent random variables with non-atomic distributions*, Trans. AMS **367** (2015), 6267–6291.
- [Voi79] D. Voiculescu, *Some results on norm-ideal perturbations of Hilbert space operators*, J. Operator Theory **2** (1979), 3–37.
- [Voi93] D. Voiculescu, *The analogues of entropy and of Fisher’s information measure in free probability theory, I*, Comm. Math. Phys. **155** (1993) 71–92.
- [Voi94] D. Voiculescu, *The analogues of entropy and of Fisher’s information measure in free probability theory, II*, Invent. Math. **118** (1994) 411–440.
- [Voi98] D. Voiculescu, *The analogues of entropy and of Fisher’s information measure in free probability, V*, Invent. Math., **132** (1998) 189–227.